# Lecture 8

**A Priori Estimates for Poisson's Equation.**

Recall that $Nf = \int_\Omega \Gamma(x-y)f(y)dy$ is called the Newtonian Potential of $f$.

**Proposition 1** *Suppose $\Omega$ is bounded domain, $f \in L^1(\Omega)$, and $\omega = Nf$ is the Newtonian potential of $f$. Then $\omega \in C^1(\mathbb{R}^n)$ and*

$$D_i\omega(x) = \int_\Omega D_i\Gamma(x-y)f(y)dy, \forall x \in \Omega.$$

**Proof:** $\Gamma = C|x|^{2-n} \Longrightarrow |D\Gamma| \leq C|x|^{1-n}$, therefore

$$v(x) = \int_\Omega D_i\Gamma(x-y)f(y)dy$$

is well defined. ($|v(x)| \leq \|f\|_{L^\infty} \int_\Omega |D_i\Gamma|dy \leq C\|f\|_{L^\infty}$.)

Define $\eta_\epsilon(t)$ to be $C^\infty$ function with properties: (1) $\eta_\epsilon(t) = 0$ for $t < \epsilon$; (2) $\eta_\epsilon(t) = 1$ for $t > 2\epsilon$; (3) $0 \leq \eta_\epsilon(t) \leq 1$; (4) $|D\eta_\epsilon| \leq \frac{2}{\epsilon}$. Define $\omega_\epsilon(x)$ to be

$$\omega_\epsilon(x) = \int_\Omega \Gamma(x-y)\eta_\epsilon(|x-y|)f(y)dy.$$

Then $\omega_\epsilon(x) \in C^1$ and $\omega_\epsilon(x) \longrightarrow \omega$ uniformly.

$$\begin{aligned}
v(x) - D_i\omega_\epsilon(x) &= \int_\Omega (D_i\Gamma(x-y) - D_i(\Gamma(x-y)\eta_\epsilon(|x-y|)))f(y)dy \\
&= \int_\Omega D_i((1-\eta_\epsilon(|x-y|))\Gamma(x-y))f(y)dy \\
&= \int_{|x-y|\leq 2\epsilon} D_i((1-\eta_\epsilon(|x-y|))\Gamma(x-y))f(y)dy
\end{aligned}$$

So

$$\begin{aligned}
|v(x) - D_i\omega_\epsilon(x)| &\leq \sup|f| \int_{|x-y|\leq 2\epsilon} (\frac{2}{\epsilon}|\Gamma(x-y)| + |D_i\Gamma(x-y)|)dy \\
&\leq \sup|f|(\frac{2}{\epsilon}\int_{|z|\leq 2\epsilon} |\Gamma(z)|dz + \int_{|z|\leq 2\epsilon} |D_i\Gamma(z)|dz) \\
&\leq \sup|f|(\frac{2}{\epsilon}\int_{|z|\leq 2\epsilon} \frac{C}{|z|^{n-2}}dz + \int_{|z|\leq 2\epsilon} \frac{C}{|z|^{n-1}}dz) \\
&\leq C\sup|f|(\frac{2}{\epsilon}\int_{r\leq\epsilon} rdr + \int_{r\leq\epsilon} dr) \\
&\leq C\cdot\epsilon\sup|f|.
\end{aligned}$$

Now we have $\omega_\epsilon \to \omega$ and $D_i\omega_\epsilon \to v$ uniformly on compact subsets as $\epsilon \to 0$, thus $\omega \in C^1(\mathbb{R}^n)$ and $D_i\omega = v$. ∎

**Theorem 1** *Let $u \in C^2(\overline{\Omega})$, $f \in L^\infty(\Omega)$ and $\Delta u = f$ in $\Omega$. Then for any compact subdomain $\Omega' \subset\subset \Omega$,*

$$\|u\|_{C^1(\Omega')} \leq C(\|u\|_{C^0(\Omega)} + \|f\|_{L^\infty(\Omega)}).$$

**Proof:** Let $\omega$ be the Newtonian potential of $f$, i.e. $\omega(y) = \int_\Omega \Gamma(x-y)f(x)dx$. Then from Green's representation formula,

$$v(y) = u(y) - \omega(y) = \int_{\partial\Omega} u(x)\frac{\partial}{\partial\nu_x}\Gamma(x-y) - \Gamma(x-y)\frac{\partial u}{\partial\nu}d\sigma_x$$

is a harmonic function. So

$$\|\omega\|_{C^0(\Omega')} = \sup_{y\in\Omega'} |\int_\Omega \Gamma(x-y)f(x)dx| \leq \|f\|_{L^\infty(\Omega)} \sup_{y\in\Omega'} \int_\Omega \frac{C}{|x-y|^{n-2}}ds \leq C\|f\|_{L^\infty(\Omega)},$$

and

$$\begin{aligned} D_i\omega(y) &= \int_\Omega D_i\Gamma(y-x)f(x)ds \\ &\leq \|f\|_{L^\infty(\Omega)} \int_\Omega D_i\Gamma(y-x)ds \\ &\leq \|f\|_{L^\infty(\Omega)} \frac{C}{|x-y|^{n-1}}ds \\ &= C\|f\|_{L^\infty(\Omega)} \end{aligned}$$

Thus $\|D_i\omega\|_{C^0(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}$, and so

$$\|\omega\|_{C^1(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}.$$

Since $v$ is harmonic, we have

$$\begin{aligned} \|Dv\|_{C^0(\Omega')} &\leq C\|v\|_{C^0(\Omega)} \\ &\leq C(\|u\|_{C^0(\Omega)} + \|\omega\|_{C^0(\Omega)}) \\ &\leq C(\|u\|_{C^0(\Omega)} + \|f\|_{L^\infty(\Omega)}). \end{aligned}$$

Thus

$$\begin{aligned} \|u\|_{C^1(\Omega')} &\leq \|v\|_{C^1(\Omega')} + \|\omega\|_{C^1(\Omega')} \\ &\leq C(\|u\|_{C^0(\Omega)} + \|f\|_{L^\infty(\Omega)}). \end{aligned}$$

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More over, one can show that for any $0 \leq \alpha < 1$,

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C(\|u\|_{C^0} + \|f\|_{L^\infty}).$$

This is not true for $\alpha = 1$.

But if $f \in C^\alpha(\overline{\Omega})$, then

$$\|u\|_{C^2(\Omega')} \le C(\|u\|_C^0 + \|f\|_{L^\infty(\Omega)})$$

and

$$\|u\|_{C^{2,\alpha}(\Omega')} \le C(\|u\|_C^\alpha + \|f\|_{L^\infty(\Omega)}).$$

**$C^{1,\alpha}$ estimate for Newtonian Potential ($\Omega$ Bounded)**

$$\omega(x) = \int_\Omega \Gamma(x-y)f(y)dy \quad \Longrightarrow \quad D_i\omega(x) = \int_\Omega D_i\Gamma(x-y)f(y)dy.$$

**Theorem 2** *If $f \in L^\infty$, then $\omega \in C^{1,\alpha}(\Omega)$.*

**Proof:** Take $x, \overline{x} \in \Omega$, let $\delta = |x - \overline{x}|$, and $\xi = \frac{1}{2}(x + \overline{x})$.

$$\begin{aligned}
D_i\omega(x) - D_i\Omega(\overline{x}) &= \int_\Omega (D_i\Gamma(x-y) - D_i\Gamma(\overline{x}-y))f(y)dy \\
&\le \|f\|_{L^\infty(\Omega)} \int_\Omega |D_i\Gamma(x-y) - D_i\Gamma(\overline{x}-y)|dy \\
&\le \|f\|_{L^\infty(\Omega)} (\int_{B_\delta(\xi)} |D_i\Gamma(x-y) - D_i\Gamma(\overline{x}-y)|dy \\
&\qquad + \int_{\Omega - B_\delta(\xi)} |D_i\Gamma(x-y) - D_i\Gamma(\overline{x}-y)|dy) \\
&= \|f\|_{L^\infty(\Omega)}(I + II),
\end{aligned}$$

where

$$\begin{aligned}
I &\le \int_{B_\delta(\xi)} |D_i\Gamma(x-y)|dy + \int_{B_\delta(\xi)} |D_i\Gamma(\overline{x}-y)|dy \\
&\le \int_{B_{\frac{3\delta}{2}}(x)} |D_i\Gamma(x-y)|dy + \int_{B_{\frac{3\delta}{2}}(x)} |D_i\Gamma(\overline{x}-y)|dy \\
&\le C \int_{B_{\frac{3\delta}{2}}(x)} \frac{1}{|x-y|^{n-1}}dy + \int_{B_{\frac{3\delta}{2}}(x)} \frac{1}{|\overline{x}-y)|^{n-1}}dy \\
&\le C \cdot \frac{3\delta}{2} \\
&= C|x - x_0|,
\end{aligned}$$

and

$$\begin{aligned}
II &= \int_{\Omega - B_\delta(\xi)} |D_i\Gamma(x-y) - D_i\Gamma(\overline{x}-y)|dy \\
&\le |x - \overline{x}| \int_{\Omega - B_\delta(\xi)} |DD_i\Gamma(\hat{x}-y)|dy \\
&\le C \cdot \delta \int_{|y-\xi| \ge \delta} \frac{1}{|\hat{x}-y|^n}dy
\end{aligned}$$

Since we have

$$\begin{aligned}
|y-\xi| &\le |y-\hat{x}| + |\hat{x}-\xi| \\
&\le |y-\hat{x}| + \frac{\delta}{2} \\
&\le |y-\hat{x}| + \frac{1}{2}|y-\xi| \\
\Longrightarrow \frac{1}{2}|y-\xi| &\le |y-\hat{x}|.
\end{aligned}$$

Thus

$$\begin{aligned}
II &\le C\cdot\delta \int_{|y-\xi|\ge\delta} \frac{1}{|y-\xi|^n}dy \\
&\le C\cdot\delta \int_\delta^R \frac{1}{r}dr \\
&\le C\cdot\delta(\log R - \log\delta) \\
&\le C\cdot\delta(\log R + c\delta^{\alpha-1}) \qquad (\because -\log\delta \le C\delta^{\alpha-1} \ for\ 0\le\alpha<1) \\
&= C\delta + C\delta^\alpha \le C\delta^\alpha \\
&= C|x-\overline{x}|^\alpha.
\end{aligned}$$

Combine the above results, we get $\omega \in C^{1,\alpha}(\Omega)$.

Further more, we get the $C^{1,\alpha}$ estimate

$$\|\omega\|_{C^{1,\alpha}(\Omega)} \le C\|f\|_{L^\infty(\Omega)},\ 0\le\alpha<1. \qquad \blacksquare$$

Just as last time, this implies

**Corollary 1** *Suppose $\Delta u = f, f\in L^\infty(\Omega), \Omega' \subset\subset \Omega$, then for $0\le\alpha<1$,*

$$\|u\|_{C^{1,\alpha}(\Omega')} \le C(\|u\|_{C^0(\Omega)} + \|f\|_{L^\infty(\Omega)}).$$

**Remark 1** *Look at the above proof and assume $f\in L^p(\Omega)$.*

$$\begin{aligned}
D_i\omega(x) - D_i\omega(\overline{x}) &= \int_\Omega (D_i\Gamma(x-y) - D_i\Gamma(\overline{x}-y))f(y)dy \\
&\le \{\int_\Omega |D_i\Gamma(x-y) - D_i\Gamma(\overline{x}-y)|^q dy\}^{1/q}\{\int_\Omega |f(y)|^p dy\}^{1/p}
\end{aligned}$$

$$\frac{1}{p}+\frac{1}{q} = 1 \Longrightarrow q = \frac{p}{p-1}.$$

In the $2^{nd}$ part of the above proof, we had

$$\begin{aligned}
II &\le \{\int_\Omega ||x-\hat{x}||DD_i\Gamma(\hat{x}-y)||^q\}^{1/q} \\
&\le C\cdot\delta(\int_{|y-\xi|\ge\delta} \frac{1}{|y-\xi|^{nq}})^{1/q} \\
&\le C\cdot\delta\cdot(\delta^{-nq+n})^{1/q} \\
&= C\cdot\delta^{1-n+\frac{n}{q}}.
\end{aligned}$$

Let $\alpha = 1 - n + n\frac{p-1}{p}$, then $p(\alpha - 1 + n) = pn - n$, i.e. $p = \frac{n}{1-\alpha}$, we have:

If $\Delta u = f, f \in L^p(\Omega), p = \frac{n}{1-\alpha}$, $\Omega' \subset\subset \Omega$, then

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C(\|f\|_{L^p(\Omega)} + \|u\|_{C^0(\Omega)}).$$

Later we will show

$$\|u\|_{W^{2,p}(\Omega')} \leq C(\|f\|_{L^p(\Omega)} + \|u\|_{C^0(\Omega)}).$$

So $C^{1,\alpha}$ estimate follows by Sobolev Embedding theorem.